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On the $\tau^{(2)}$-model in the chiral Potts model and cyclic representation of the quantum group $U_q(sl_2)$

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Abstract

We identify the precise relationship between the five-parameter $\tau^{(2)}$-family in the N-state chiral Potts model and XXZ chains with $U_q(sl_2)$-cyclic representation. By studying the Yang–Baxter relation of the six-vertex model, we discover a one-parameter family of L-operators in terms of the quantum group $U_q(sl_2)$. When N is odd, the N-state $\tau^{(2)}$-model can be regarded as the XXZ chain of $U_q(sl_2)$ cyclic representations with $q^N = 1$. The symmetry algebra of the $\tau^{(2)}$-model is described by the quantum affine algebra $U_q(\widehat{sl}_2)$ via the canonical representation. In general, for an arbitrary N, we show that the XXZ chain with a $U_q(sl_2)$-cyclic representation for $q^{2N} = 1$ is equivalent to two copies of the same N-state $\tau^{(2)}$-model.

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1. Introduction

In the study of the N-state chiral Potts model (CPM), Bazhanov and Stroganov [12] discovered a five-parameter family of L-operators of the $\tau^{(2)}$-model in the six-vertex model with a particular field (see [9] page 3), i.e. the solution of Yang–Baxter (YB) equation for the *asymmetric* six-vertex R-matrix, (see (2.1) and (2.2) in this paper). The chiral Potts transfer matrix can be constructed as the Baxter's Q-operator of the $\tau^{(2)}$-model [11, 29]. By the functional relations between the fusion matrices of the $\tau^{(2)}$-model and the chiral Potts transfer matrix, one can compute the eigenvalue spectrum of the (homogeneous) superintegrable CPM [1, 6, 7], where the $\tau^{(2)}$-degeneracy occurs with the symmetry algebra described by Onsager algebra [24]. Furthermore, one can discuss the eigenvalue spectrum and calculate the order parameter in the CPM through the functional-relation method [4, 5, 10, 22, 30]. Though much progress has been made on issues related to the eigenvalues, we still lack enough information at present about the eigenvectors in the CPM. Consequently, some important problems

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which require knowledge of eigenvectors, such as the calculation of correlation functions, remain unsolved in the theory. On the other hand, the ('zero-field') six-vertex model ([8] section 3) with the symmetric R-matrix has been a well-studied theory with profound knowledge known in the literature about their eigenvalues and eigenvectors. The understanding of the structure was further extended to XXZ models of higher spin [20], or more generally the model associated with a representation of the quantum group $U_q(sl_2)$. When q is a Nth root of unity, it was shown in [15–17, 23, 25, 27] that the degeneracy of XXZ models occurs with the extra sl_2-loop-algebra symmetry induced from the structure of the quantum affine algebra $U_q(\widehat{sl_2})$. For odd N, as a special member of a one-parameter $U_q(sl_2)$-cyclic representation, the spin-$(N-1)/2$ XXZ chain can be identified with the superintegrable N-state $\tau^{(2)}$-model [28]. Hence the superintegrable $\tau^{(2)}$-model also carries the sl_2-loop-algebra symmetry for the degeneracy (of certain sectors), compatible with the Onsager-algebra symmetry inherited from the chiral Potts transfer matrix [28]. The interrelation of these two symmetries can lead to some useful information about eigenvectors of the superintegrable model (see, e.g. [2, 3]). The aim of this paper is to extend the equivalent relationship between the N-state $\tau^{(2)}$-model and the XXZ model with the $U_q(sl_2)$-cyclic representation to the whole five-parameter $\tau^{(2)}$-family found in [12] for any N. Through the YB relation in the six-vertex model, we find a characterization of a quantum group $U_q(sl_2)$ through a one-parameter L-operators of the YB solution (see (2.5), (2.6) in the paper). When q is a root of unity, the algebra $U_q(sl_2)$ possesses a three-parameter family of cyclic representation. Together with the rescaling of the spectral parameter, XXZ chains with the $U_q(sl_2)$-cyclic representation also carry a five-parameter L-operator. By extending the argument in [28], we show the identification of $\tau^{(2)}$-models and the XXZ chains with the $U_q(sl_2)$-cyclic representation. Note that such a precise connection between the $\tau^{(2)}$-family in the CPM and the XXZ-family with the symmetric R-matrix seems not to have appeared in the literature before, to the best of the author's knowledge, even in the odd N case except a one-parameter family of $\tau^{(2)}$-models in [28]. We hope an identification of these models will help to provide some useful clues to understand the eigenvectors of the CPM in certain special cases.

This paper is organized as follows. In section 2, we recall the definition of the $\tau^{(2)}$-model, quantum group $U_q(sl_2)$, and the quantum affine algebra $U_q(\widehat{sl_2})$. In section 3, we first describe the three-parameter family of cyclic representations of $U_q(sl_2)$ for a root of unity q. The case of $\mathsf{q}^N = 1$ for odd N is discussed in subsection 3.1 where the $\tau^{(2)}$-model is identified with the XXZ chain with cyclic representation of $U_\mathsf{q}(sl_2)$, hence the quantum space carries a canonical representation of the quantum affine algebra $U_\mathsf{q}(\widehat{sl_2})$. In subsection 3.2, we study the N-state $\tau^{(2)}$-model for an arbitrary N. We show that a XXZ chain of $U_q(sl_2)$-cyclic representation with $q^{2N} = 1$ is equivalent to two copies of the same N-state $\tau^{(2)}$-model. Finally we close in section 4 with a brief concluding remark.

Notation: we use standard notations. For a positive integer N greater than one, $\mathbf{C}^N$ denotes the vector space of N-cyclic vectors with the canonical base $|n\rangle$, $n \in \mathbf{Z}_N (:= \mathbf{Z}/N\mathbf{Z})$. We fix the Nth root of unity $\omega = e^{\frac{2\pi i}{N}}$, and X, Z, the Weyl $\mathbf{C}^N$-operators

$$X|n\rangle = |n+1\rangle, \qquad Z|n\rangle = \omega^n|n\rangle \quad (n \in \mathbf{Z}_N),$$

which satisfy $X^N = Z^N = 1$ and the Weyl relation: $XZ = \omega^{-1}ZX$.

2. The N-state $\tau^{(2)}$-model and quantum group $U_q(sl_2)$

The N-state $\tau^{(2)}$-model [11, 12, 30] (also called the Baxter–Bazhanov–Stroganov model [19]) is the five-parameter family of L-operators of $\mathbf{C}^2$-auxiliary, $\mathbf{C}^N$-quantum space with entries expressed by Weyl operators X, Z

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$$L(t) = \begin{pmatrix} 1 - t\dfrac{\mathsf{c}}{\mathsf{b'b}}X & \left(\dfrac{1}{\mathsf{b}} - \omega\dfrac{\mathsf{ac}}{\mathsf{b'b}}X\right)Z \\ -t\left(\dfrac{1}{\mathsf{b'}} - \dfrac{\mathsf{a'c}}{\mathsf{b'b}}X\right)Z^{-1} & -t\dfrac{1}{\mathsf{b'b}} + \omega\dfrac{\mathsf{a'ac}}{\mathsf{b'b}}X \end{pmatrix}, \tag{2.1}$$

where t is the spectral variable, and $\mathsf{a}, \mathsf{b}, \mathsf{a'}, \mathsf{b'}, \mathsf{c}$ are nonzero complex parameters. It is known that the L-operator (2.1) satisfies the YB equation

$$R(t/t')\big(\mathsf{L}(t)\bigotimes_{\text{aux}} 1\big)\big(1\bigotimes_{\text{aux}} \mathsf{L}(t')\big) = \big(1\bigotimes_{\text{aux}} \mathsf{L}(t')\big)\big(\mathsf{L}(t)\bigotimes_{\text{aux}} 1\big)R(t/t')$$

for the asymmetry six-vertex R-matrix

$$R(t) = \begin{pmatrix} t\omega - 1 & 0 & 0 & 0 \\ 0 & t - 1 & \omega - 1 & 0 \\ 0 & t(\omega - 1) & (t - 1)\omega & 0 \\ 0 & 0 & 0 & t\omega - 1 \end{pmatrix}. \tag{2.2}$$

The monodromy matrix, $\bigotimes_{\ell=1}^{L} \mathsf{L}_\ell(t)$ where $\mathsf{L}_\ell(t) = \mathsf{L}(t)$ at site ℓ, again satisfies the above YB relation, whose ω-twisted trace defines the $\tau^{(2)}$-matrix

$$\tau^{(2)}(t) = \text{tr}_{\mathbf{C}^2} \bigotimes_{\ell=1}^{L} \mathsf{L}_\ell(\omega t), \tag{2.3}$$

commuting with the spin-shift operator $X\big(:= \prod_\ell X_\ell\big)$.

The quantum group $U_q(sl_2)$ is the associated **C**-algebra generated by $K^{\frac{\pm 1}{2}}, e^\pm$ with the relations $K^{\frac{1}{2}} K^{\frac{-1}{2}} = K^{\frac{-1}{2}} K^{\frac{1}{2}} = 1$ and

$$K^{\frac{1}{2}} e^\pm K^{\frac{-1}{2}} = q^{\pm 1} e^\pm, \qquad [e^+, e^-] = \frac{K - K^{-1}}{q - q^{-1}}. \tag{2.4}$$

Then the two-by-two matrix with $U_q(sl_2)$-entries defines a one-parameter family of L-operators

$$\mathcal{L}(s) = \begin{pmatrix} \rho^{-1} s K^{\frac{-1}{2}} - s^{-1} K^{\frac{1}{2}} & (q - q^{-1}) e^- \\ (q - q^{-1}) e^+ & s K^{\frac{1}{2}} - \rho s^{-1} K^{\frac{-1}{2}} \end{pmatrix}, \qquad \rho \neq 0 \in \mathbf{C}, \tag{2.5}$$

which satisfy the YB equation

$$R_{6v}(s/s')(\mathcal{L}(s)\bigotimes_{\text{aux}} 1)\big(1\bigotimes_{\text{aux}} \mathcal{L}(s')\big) = \big(1\bigotimes_{\text{aux}} \mathcal{L}(s')\big)\big(\mathcal{L}(s)\bigotimes_{\text{aux}} 1\big)R_{6v}(s/s'). \tag{2.6}$$

for the six-vertex (symmetric) R-matrix [18, 21]:

$$R_{6v}(s) = \begin{pmatrix} s^{-1}q - sq^{-1} & 0 & 0 & 0 \\ 0 & s^{-1} - s & q - q^{-1} & 0 \\ 0 & q - q^{-1} & s^{-1} - s & 0 \\ 0 & 0 & 0 & s^{-1}q - sq^{-1} \end{pmatrix}.$$

Indeed, the YB relation (2.6) for $\mathcal{L}$ in (2.5) is the necessary and sufficient condition of the constraint (2.4) in the definition of quantum group $U_q(sl_2)$. Using the local L-operator (2.5), one constructs the monodromy matrix $\bigotimes_{\ell=1}^{L} \mathcal{L}_\ell(s)$ with entries in $\big(\overset{L}{\otimes} U_q(sl_2)\big)(s)$,

$$\bigotimes_{\ell=1}^{L} \mathcal{L}_\ell(s) = \begin{pmatrix} \mathcal{A}_L(s) & \mathcal{B}_L(s) \\ \mathcal{C}_L(s) & \mathcal{D}_L(s) \end{pmatrix}$$

again satisfying (2.6). The structure of the quantum affine algebra $U_q(\widehat{sl}_2)$ is described by the leading and lowest terms of the above entries in the monodromy matrix,

$$\mathsf{A}_+ = \lim_{s\to\infty}(\rho^{-1}s)^{-L}\mathcal{A}_L(s), \qquad\qquad \mathsf{A}_- = \lim_{s\to 0}(-s)^L\mathcal{A}_L(s),$$

$$\mathsf{B}_\pm = \lim_{s^{\pm 1}\to\infty}(\pm s)^{\mp(L-1)}\frac{\mathcal{B}_L(s)}{q-q^{-1}}, \qquad \mathsf{C}_\pm = \lim_{s^{\pm 1}\to\infty}(\pm s)^{\mp(L-1)}\frac{\mathcal{C}_L(s)}{q-q^{-1}},$$

$$\mathsf{D}_+ = \lim_{s\to\infty}s^{-L}\mathcal{D}_L(s), \qquad\qquad \mathsf{D}_- = \lim_{s\to 0}(-\rho^{-1}s)^L\mathcal{D}_L(s).$$

One has $\mathsf{A}_+ = \mathsf{A}_-^{-1}, \mathsf{A}_\mp = \mathsf{D}_\pm$. Denote $T^- = \mathsf{B}_+, S^- = \mathsf{B}_-$ and $S^+ = \mathsf{C}_+, T^+ = \mathsf{C}_-$. The operators

$$k_0^{-1} = k_1 = \mathsf{A}_-^2 = \mathsf{D}_+^2, \qquad e_1 = S^+, \qquad f_1 = S^-, \qquad e_0 = T^-, \qquad f_0 = T^+$$

generate the quantum affine algebra $U_q(\widehat{sl}_2)$ with the Hopf-algebra structure

$$\triangle(k_i) = k_i \otimes k_i, \qquad i = 0, 1,$$
$$\triangle(e_1) = k_1 \otimes e_1 + \rho^{-1}e_1 \otimes k_0, \qquad \triangle(f_1) = k_1 \otimes f_1 + \rho f_1 \otimes k_0,$$
$$\triangle(e_0) = \rho^{-1}k_0 \otimes e_0 + e_0 \otimes k_1, \qquad \triangle(f_0) = \rho k_0 \otimes f_0 + f_0 \otimes k_1.$$

Indeed, the explicit expression of generators of $U_q(\widehat{sl}_2)$ is

$$\mathsf{A}_- = K^{\frac{1}{2}} \otimes \cdots \otimes K^{\frac{1}{2}}, \qquad \mathsf{A}_+ = K^{\frac{-1}{2}} \otimes \cdots \otimes K^{\frac{-1}{2}},$$

$$S^\pm = \sum_{i=1}^{L} \underbrace{K^{\frac{1}{2}} \otimes \cdots \otimes K^{\frac{1}{2}}}_{i-1} \otimes e^\pm \otimes \underbrace{K^{\frac{-1}{2}} \otimes \cdots \otimes K^{\frac{-1}{2}}}_{L-i} \rho^{\mp(L-i)}, \qquad (2.7)$$

$$T^\pm = \sum_{i=1}^{L} \rho^{\pm(i-1)} \underbrace{K^{\frac{-1}{2}} \otimes \cdots \otimes K^{\frac{-1}{2}}}_{i-1} \otimes e^\pm \otimes \underbrace{K^{\frac{1}{2}} \otimes \cdots \otimes K^{\frac{1}{2}}}_{L-i}.$$

Note that (2.6) is still valid when changing the variable s by αs for a nonzero complex α. Given a finite-dimensional representation $\sigma : U_q(sl_2) \longrightarrow \text{End}(\mathbf{C}^d)$, the L-operator of $\mathbf{C}^2$-auxiliary and $\mathbf{C}^d$-quantum space, $L(s) = \sigma(\mathcal{L}(\alpha s))$, satisfies the YB relation (2.6). In particular, by setting the parameter $\rho = 1$ in (2.5), with $\alpha = q^{\frac{d-2}{2}}$ and σ the spin-$\frac{d-1}{2}$ (highest-weight) representation of $U_q(sl_2)$ on $\mathbf{C}^d = \oplus_{k=0}^{d-1}\mathbf{C}\,\mathbf{e}^k$

$$K^{\frac{1}{2}}(\mathbf{e}^k) = q^{\frac{d-1-2k}{2}}\,\mathbf{e}^k, \qquad e^+(\mathbf{e}^k) = [k]\,\mathbf{e}^{k-1}, \qquad e^-(\mathbf{e}^k) = [d-1-k]\,\mathbf{e}^{k+1},$$

where $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ and $e^+(\mathbf{e}^0) = e^-(\mathbf{e}^{d-1}) = 0$, one obtains the well-known L-operator of the XXZ chain of spin-$\frac{d-1}{2}$ (see, e.g. [20, 26, 27] and references therein).

3. The equivalence of $\tau^{(2)}$-models and XXZ chains with cyclic representation of $U_q(sl_2)$

In this section, we consider the case when q is a root of unity. For a Nth root of unity q, there exists a three-parameter family of cyclic representation $\sigma_{\phi,\phi',\varepsilon}$ of $U_q(sl_2)$, labeled by nonzero complex numbers ϕ, ϕ' and ε, which acts on cyclic $\mathbf{C}^N$-vectors by

$$K^{\frac{1}{2}}|n\rangle = q^{-n+\frac{\phi'-\phi}{2}}|n\rangle, \qquad e^+|n\rangle = q^\varepsilon\frac{q^{\phi+n} - q^{-\phi-n}}{q - q^{-1}}|n-1\rangle,$$

$$e^-|n\rangle = q^{-\varepsilon}\frac{q^{\phi'-n} - q^{-\phi'+n}}{q - q^{-1}}|n+1\rangle, \qquad (3.1)$$

(see, e.g. [13, 14]). The L-operator,

$$L(s) = \sigma_{\phi,\phi',\varepsilon}\mathcal{L}(s) \qquad (3.2)$$

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gives rise to the transfer matrix of the XXZ chain with the $U_q(sl_2)$-cyclic representation $\sigma_{\phi,\phi',\varepsilon}$,

$$T(s) = (\otimes \sigma_{\phi,\phi',\varepsilon}) \left(\mathrm{tr}_{\mathbf{C}^2} \bigotimes_{\ell=1}^{L} \mathcal{L}_\ell(s) \right), \tag{3.3}$$

which commutes with $K^{\frac{1}{2}} \left(:= \otimes_\ell K_\ell^{\frac{1}{2}}, \text{ the product of local } K^{\frac{1}{2}}\text{-operators}\right)$.

3.1. The identification of the N-state $\tau^{(2)}$-model and the XXZ chain of $U_q(sl_2)$-cyclic representation with $q^N = 1$ for odd N

In the subsection, we consider the case N odd, and write $N = 2M + 1$. Let q be the primitive Nth root-of-unity with $q^{-2} = \omega$. Then one can express the cyclic representation (3.1) in terms of the Weyl operators X, Z

$$K^{\frac{1}{2}} = q^{\frac{\phi'-\phi}{2}} Z^{\frac{1}{2}}, \qquad e^+ = q^\varepsilon \frac{\left(q^{\phi+1} Z^{-\frac{1}{2}} - q^{-\phi-1} Z^{\frac{1}{2}}\right) X^{-1}}{q - q^{-1}},$$

$$e^- = q^{-\varepsilon} \frac{\left(q^{\phi'+1} Z^{\frac{1}{2}} - q^{-\phi'-1} Z^{-\frac{1}{2}}\right) X}{q - q^{-1}},$$

hence follow the expression:

$$K^{-1} = q^{\phi-\phi'} Z^{-1}, \qquad K^{\frac{-1}{2}} e^+ = -q^{\frac{-\phi-\phi'}{2}+\varepsilon-1} \frac{(1 - q^{2\phi+2} Z^{-1}) X^{-1}}{q - q^{-1}},$$

$$K^{\frac{-1}{2}} e^- = q^{\frac{\phi+\phi'}{2}-\varepsilon+1} \frac{(1 - q^{-2\phi'-2} Z^{-1}) X}{q - q^{-1}}. \tag{3.4}$$

Using the Fourier basis $\widehat{|n\rangle} = \frac{1}{N} \sum_{j \in \mathbf{Z}_N} \omega^{-nj} |j\rangle$, one may convert the Weyl operator (Z^{-1}, X) to (X, Z)

$$\left({}_X^{Z^{-1}} \widehat{|0\rangle}, \ldots, {}_X^{Z^{-1}} \widehat{|N-1\rangle} \right) = \left(\widehat{|0\rangle}, \ldots, \widehat{|N-1\rangle} \right)_Z^X, \tag{3.5}$$

hence represent $K^{-1}, K^{\frac{-1}{2}} e^\pm$ in (3.4) by

$$K^{-1} = q^{\phi-\phi'} X, \qquad K^{\frac{-1}{2}} e^+ = -q^{\frac{-\phi-\phi'}{2}+\varepsilon-1} \frac{(1 - q^{2\phi+2} X) Z^{-1}}{q - q^{-1}},$$

$$K^{\frac{-1}{2}} e^- = q^{\frac{\phi+\phi'}{2}-\varepsilon+1} \frac{(1 - q^{-2\phi'-2} X) Z}{q - q^{-1}}.$$

By the above expression and introducing the spectral variable $t = \lambda^{-1} s^2$ for a nonzero complex λ, the modified L-operator of (3.2), $-s K^{\frac{-1}{2}} L(s)$, is gauge equivalent to

$$\begin{pmatrix} 1 - t\lambda\rho^{-1} q^{\phi-\phi'} X & q^{-\varepsilon}(1 - q^{-2\phi'-2} X) Z \\ -t\lambda q^\varepsilon (1 - q^{2\phi+2} X) Z^{-1} & -t\lambda + \rho q^{\phi-\phi'} X \end{pmatrix},$$

which is the same as the L-operator (2.1) of the $\tau^{(2)}$-model by the following identification of parameters:

$$a = \lambda^{-1} \rho q^{-\phi-\phi'-\varepsilon}, \quad a' = \rho q^{\phi+\phi'+\varepsilon+2}, \quad b = q^\varepsilon, \quad b' = \lambda^{-1} q^{-\varepsilon}, \quad c = \rho^{-1} q^{\phi-\phi'},$$

equivalently $\quad q^\varepsilon = b, \quad q^{2\phi} = \frac{\omega a' c}{b}, \quad q^{2\phi'} = \frac{b'}{ac}, \quad \rho^2 = \frac{\omega a a'}{bb'}, \quad \lambda = \frac{1}{bb'}. \tag{3.6}$$

This implies the transfer matrix (3.3) of the XXZ chain is equivalent to the $\tau^{(2)}$-transfer matrix (2.3). Note that the product of the local operator K^{-1}'s of the XXZ chain is now corresponding to a scalar multiple of the spin-shift operator X, which commutes with the $\tau^{(2)}$-matrix. The

representations with $\varepsilon = 0$, $\phi = \phi'$ in (3.1) form the one-parameter cyclic representation of $U_q(sl_2)$ discussed in [28, section 4]. In particular, the case $\phi = \phi' = M$ is the spin-$\frac{N-1}{2}$ highest-weight representation of $U_q(sl_2)$.

It is known that the degeneracy of the $\tau^{(2)}$-model in the CPM occurs in the alternating superintegrable case (see, [30] section 4.3), where the $\tau^{(2)}$-model is characterized by the L-operator (2.1) with the parameter

$$\mathsf{a} = \omega^{\mathsf{m}} \mathsf{b}', \qquad \mathsf{a}' = \omega^{\mathsf{m}'} \mathsf{b}, \qquad \mathsf{c} = \omega^{\mathsf{n}}, \qquad (\mathsf{m}, \mathsf{m}', \mathsf{n} \in \mathbf{Z}),$$

or equivalently, with the representation parameters ϕ, ϕ', ε and ρ, λ in (3.6) given by

$$\phi = -(\mathsf{m}' + \mathsf{n} + 1), \qquad \phi' = \mathsf{m} + \mathsf{n}, \qquad \mathsf{q}^{\varepsilon} = \mathsf{b}, \qquad \rho = \mathsf{q}^{-(1+\mathsf{m}+\mathsf{m}')}, \qquad \lambda = \frac{1}{\mathsf{b}\mathsf{b}'}. \tag{3.7}$$

In this case, there exist the normalized Nth power of $S^{\pm}$, $T^{\pm}$ in (2.7), $S^{\pm(N)} = \frac{S^{\pm N}}{[N]!}$, $T^{\pm(N)} = \frac{S^{\pm N}}{[N]!}$. As ϕ in (3.7) is an integer, by the algebraic-Bethe-ansatz technique, one can show the degeneracy of the alternating superintegrable $\tau^{(2)}$-model (for certain sectors) possesses the symmetry algebra generated by $S^{\pm(N)}$, $T^{\pm(N)}$, which can be identified with the sl_2-loop algebra under certain constraints between the integers m, m' and n, as in the discussion of the homogeneous superintegrable case in [15, 23, 27]. However, the precise structure about the symmetry algebra for arbitrary m, m', n remains unknown to be identified.

3.2. The XXZ chain of the $U_q(sl_2)$-cyclic representation with $q^{2N} = 1$ as two copies of the N-state $\tau^{(2)}$-model for an arbitrary N

By extending the argument of the previous subsection, we now identify the N-state $\tau^{(2)}$-model with a XXZ chain for an arbitrary N. Let q be a primitive $(2N)$-th root of unity with $q^{-2} = \omega$, hence $q^N = -1$. Consider the three-parameter family of $U_q(sl_2)$-cyclic representations $\sigma_{\phi,\phi',\varepsilon}$ in (3.1) on the vector space V of $2N$-cyclic vectors

$$V := \bigoplus \{\mathbf{C}|n\rangle' \,|\, n \in \mathbf{Z}_{2N}\}.$$

Denote the vectors in V

$$|n\rangle := |n\rangle' + |n + N\rangle', \qquad (n \in \mathbf{Z}_{2N}),$$
$$|n\rangle_- := |n\rangle' - |n + N\rangle', \qquad (0 \leqslant n \leqslant N - 1).$$

Then $|n\rangle = |n + N\rangle$. We define $|n\rangle_-$ for $n \in \mathbf{Z}$ by the N-periodic condition: $|n + N\rangle_- = |n\rangle_-$. Hence we can identify $\{|n\rangle\}$ or $\{|n\rangle_-\}$ with the canonical basis of the vector space of N-cyclic vectors, and one has the decomposition of V

$$V = V_+ \oplus V_-, \qquad V_+ := \sum_{n \in \mathbf{Z}_N} \mathbf{C}|n\rangle, \qquad V_- := \sum_{n \in \mathbf{Z}_N} \mathbf{C}|n\rangle_-.$$

The spin-shift operator $|n\rangle' \mapsto |n + 1\rangle'$ of V is decomposed as the sum of X on V_+ and $\widetilde{X}$ on V_-, where

$$X|n\rangle = |n + 1\rangle (n \in \mathbf{Z}_N), \qquad \widetilde{X}|n\rangle_- = \begin{cases} |n + 1\rangle_- & n \not\equiv N - 1 \pmod{N}, \\ -|n + 1\rangle_- & n \equiv N - 1 \pmod{N}. \end{cases}$$

Together with the operator Z: $Z|n\rangle = \omega^n|n\rangle$, $Z|n\rangle_- = \omega^n|n\rangle_-$, one finds (X, Z), $(\widetilde{X}, Z)$ are Weyl operators of V_+ or V_- respectively with $X^N = Z^N = 1$, $\widetilde{X}^N = -1$. Under the

cyclic representation $\sigma_{\phi,\phi',\varepsilon}$ (3.1) of $U_q(sl_2)$, the subspaces V_+, V_- are interchanged under the generators $K^{\frac{1}{2}}, e^{\pm}$, with the expression

$$K^{\frac{1}{2}}|n\rangle = q^{-n+\frac{\phi'-\phi}{2}}|n\rangle_-, \qquad K^{\frac{1}{2}}|n\rangle_- = q^{-n+\frac{\phi'-\phi}{2}}|n\rangle \quad (0 \leqslant n \leqslant N-1),$$

$$e^+|n\rangle = q^{\varepsilon}\frac{q^{\phi+n}-q^{-\phi-n}}{q-q^{-1}}|n-1\rangle_- \quad (1 \leqslant n \leqslant N),$$

$$e^-|n\rangle = q^{-\varepsilon}\frac{q^{\phi'-n}-q^{-\phi'+n}}{q-q^{-1}}|n+1\rangle_- \quad (-1 \leqslant n \leqslant N-2), \tag{3.8}$$

$$e^+|n\rangle_- = q^{\varepsilon}\frac{q^{\phi+n}-q^{-\phi-n}}{q-q^{-1}}|n-1\rangle,$$

$$e^-|n\rangle_- = q^{-\varepsilon}\frac{q^{\phi'-n}-q^{-\phi'+n}}{q-q^{-1}}|n+1\rangle \quad (0 \leqslant n \leqslant N-1).$$

Using the above formulae, one finds that K^{-1} and $K^{\frac{-1}{2}}e^{\pm}$ are operators of V_+ and V_-, which can be expressed by X, Z and $\widetilde{X}, Z$ respectively such that relation (3.4) (using q) holds for V_+, and also valid for V_- when replacing X by $\widetilde{X}$. By using the V_--basis, $|n\rangle\rangle := q^n|n\rangle_-$ for $0 \leqslant n \leqslant N-1$, the Weyl pair $(\widetilde{X}, Z)$ is converted to $(q^{-1}X, Z)$. One can express the operators $K^{\frac{-1}{2}}e^{\pm}$ by

$$K^{\frac{-1}{2}}e^+|n\rangle\rangle = -q^{\frac{-\phi-\phi'}{2}+\varepsilon}\frac{(1-q^{2\phi+2}Z^{-1})X^{-1}}{q-q^{-1}}|n\rangle\rangle,$$

$$K^{\frac{-1}{2}}e^-|n\rangle\rangle = q^{\frac{\phi+\phi'}{2}-\varepsilon}\frac{(1-q^{-2\phi'-2}Z^{-1})X}{q-q^{-1}}|n\rangle\rangle.$$

Using the Fourier transform (3.5) of the basis $\{|n\rangle\}$ and $\{|n\rangle\rangle\}$, by the same argument in subsection 3.1, one can show that $-sK^{\frac{-1}{2}}L(s)$ for the L-operator (3.2) is gauge equivalent to two copies of the same L-operator (2.1) of the $\tau^{(2)}$-model with the parameter given by (3.6) (replacing q by q). Therefore the transfer matrix (3.3) of the XXZ chain is equivalent to the sum of two copies of the $\tau^{(2)}$-transfer matrix (2.3). Note that neither one of the two copies of the $\tau^{(2)}$-model inherits the $U_q(sl_2)$ structure of the XXZ chain, in which one copy is sent to another by (3.8). The symmetry structure of the $\tau^{(2)}$-model related to $U_q(sl_2)$ is different from that in subsection 3.1 for the odd N case. Nevertheless, the $\tau^{(2)}$-model with certain constraints on the parameter again possesses the sl_2-loop-algebra symmetry that arises from the $U_q(\widehat{sl_2})$-structure of the corresponding XXZ-model. In particular, in the homogeneous superintegrable case, the Onsager-algebra symmetry of the degenerate $\tau^{(2)}$-eigenspace is extended to the symmetry of sl_2-loop algebra (in certain sectors). The relationship of these two symmetries will hopefully, though not immediately apparent, lead to a solution of the eigenvector problem in the superintegrable CPM (for an arbitrary N) along the line in [2, 3].

4. Concluding remarks

By studying the general solution of the YB equation of the six-vertex model, we find a one-parameter L-operators associated with the quantum group $U_q(sl_2)$, which carries a three-parameter family of cyclic representation when q is a root of unity. We have showed that the XXZ chain of the $U_q(sl_2)$-cyclic representation with $q^{2N} = 1$ is equivalent to the sum of two copies of a N-state $\tau^{(2)}$-model. In particular, the Onsager-algebra symmetry of the homogeneous superintegrable CPM is enlarged to the sl_2-loop-algebra symmetry induced from the corresponding XXZ-model. When N is odd and q is a primitive Nth root of unity, the N-state $\tau^{(2)}$-model can also be identified with the XXZ chain of a cyclic representation

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of $U_q(sl_2)$, hence the quantum space carries the structure of quantum affine algebra $U_q(\widehat{sl_2})$. As a special case, the homogeneous superintegrable $\tau^{(2)}$-model in the CPM is equivalent to the spin-$(N-1)/2$ XXZ chain with the canonical $U_q(\widehat{sl_2})$-structure [28], by which the sl_2-loop-algebra symmetry of the superintegrable $\tau^{(2)}$-model was derived in [23, 27]. In the recent study of eigenvalues of the chiral Potts model with alternating rapidities [30], the $\tau^{(2)}$-degeneracy is also found in the alternating superintegrable case. By identifying these $\tau^{(2)}$-models with the XXZ chains via (3.7), we find the sl_2-loop-algebra symmetry for the degenerate $\tau^{(2)}$-eigenspace in certain cases. Similar to the homogeneous CPM case [24], some twisted version of the Onsager-algebra symmetry for the $\tau^{(2)}$-model could also possibly exist in the theory of a general alternating superintegrable chiral Potts model. However, the explicit nature remains hard to know precisely. The exact relationship and a further symmetry study of general cases are now under consideration.

Acknowledgments

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